

# Non-Perturbative Improvement of the Anisotropic Wilson QCD Action\*

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We describe the first steps in the extension of the Symanzik  $O(a)$  improvement program for Wilson-type quark actions to *anisotropic* lattices, with a temporal lattice spacing smaller than the spatial one. This provides a fully relativistic and computationally efficient framework for the study of heavy quarks. We illustrate our method with accurate results for the quenched charmonium spectrum.

The simulation of charm or bottom quarks on (a sequence of) lattices with  $am_q \ll 1$  is currently not affordable, even in the quenched approximation. Since standard lattice actions break down when  $am_q > 1$ , one must design special actions to study heavy quarks. Currently two approaches are widely used: (i) Non-relativistic lattice QCD (NRQCD) [1], and (ii) the “heavy relativistic” or “Fermilab” approach [2].

The heavy relativistic approach aims to implement the improvement program for the Wilson action at any quark mass. Then the action only has manifest  $O(3)$  symmetry, and to improve it to a given order in the spatial momenta requires all coefficients to be *mass-dependent*. The approach we would like to advocate [3,4] can be denoted as “anisotropic relativistic”, for short. It is similar to the heavy relativistic one in that the same terms have to be included in the lattice action, but it avoids the mass dependence of the coefficients (except for one, see below) by using a temporal lattice spacing  $a_0$  so small that  $a_0 m_q \ll 1$ , and aiming to improve the action fully only up to  $O(a)$ . It has become clear in the last few years that a *non-perturbative* determination of the leading improvement coefficients in a Wilson-type action is very important [5,6]. In our approach this can be achieved, we believe, but in others the large number and/or mass-dependence of the coefficients make this appear very difficult.

Consider an anisotropic lattice with fixed (renormalized) anisotropy  $\xi = a/a_0$ . Up to  $O(a)$  a Wilson-type quark action on such a lattice can

have the following terms, in continuum notation,

$$\begin{aligned} & \not{D}_0 (1 + O(am)) , \quad \not{D} (1 + O(am)) , \\ & a D_0^2, \quad a D_k^2, \quad a [\not{D}, \not{D}_0], \quad a \sigma_{0k} F_{0k}, \quad a \sigma_{kl} F_{kl} \end{aligned} \quad (1)$$

Here  $\sigma_{\mu\nu} = -\frac{i}{2}[\gamma_\mu, \gamma_\nu]$  and  $F_{\mu\nu}$  is the field strength. Using a field transformation we can set the coefficient of  $\not{D}_0$  (say) to 1 and adjust the coefficients of the first three terms in the second line of (1). We set the third term to zero, and adjust the second-order derivatives to combine with the first-order ones to give the computationally efficient Wilson operators,  $W_\mu \equiv \nabla_\mu - \frac{1}{2}a_\mu \Delta_\mu$ , where  $\nabla_\mu$  and  $\Delta_\mu$  are the standard discretizations of first- and second-order covariant derivatives (other possible strategies are discussed in [4]). This gives a quark matrix  $Q$  of the form

$$\begin{aligned} Q = & m_0 + W_0 \gamma_0 + \nu \sum_k W_k \gamma_k \\ & - \frac{a}{2} \left[ \omega_0 \sum_k \sigma_{0k} F_{0k} + \omega \sum_{k<l} \sigma_{kl} F_{kl} \right]. \end{aligned} \quad (2)$$

Here  $\nu$  is a “bare velocity of light” and  $\omega_0, \omega$  are the temporal and spatial “clover coefficients”; all three have to be tuned on the quantum level.

It is instructive to compare our and the heavy relativistic approach. The latter corresponds to the special case of an isotropic lattice,  $\xi = 1$ , and one must adjust the coefficients  $\nu, \omega_0, \omega$  to all orders in the mass. As an example consider the classical tuning of  $\nu$ . Defining an “effective velocity of light”  $c(\mathbf{p})$  via the dispersion relation  $E(\mathbf{p})^2 = E(0)^2 + c(\mathbf{p})^2 \mathbf{p}^2$ , we demand  $c(0) = 1$  for a free quark. The result for  $\nu$  is shown in figure 1 for various  $\xi$ , as a function of the pole mass  $m = \ln(1 + a_0 m_0)/a_0$  of the quark. Note that

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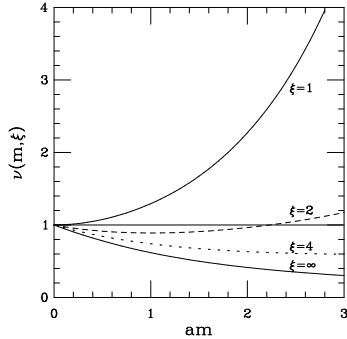


Figure 1. Classical bare velocity of light.

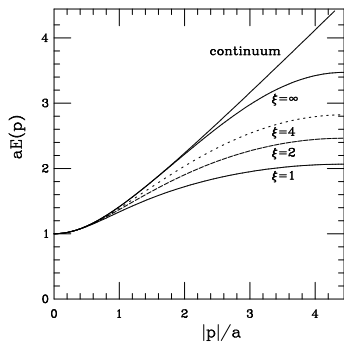


Figure 2. Dispersion relation  $E(\mathbf{p})$  of a free, massive quark for  $\mathbf{p} \propto (1, 1, 0)$  after tuning  $\nu$ .

$\nu$  has a strong mass dependence in the isotropic case, but only a very weak one for anisotropic lattices. The dispersion relation is shown in figure 2. It is much better for anisotropic lattices, illustrating that they are the natural habitat for heavy quarks.

One can also work out the exact classical mass dependence of  $\omega_0$  and  $\omega$ . It is (almost) identical to figure 1, with a strong/weak mass dependence in the isotropic/anisotropic cases. In our approach we can choose  $\omega_0$  and  $\omega$  to be mass-independent, but must adjust  $\nu$  to  $O(a)$  (for details and subtleties see [4]). We can also tune  $\nu$  non-perturbatively, by demanding that the pseudo-scalar meson, say, satisfy  $c(0)=1$ . In fact, it is *simpler* to determine  $\nu$  exactly for the heavy masses of interest in this manner; we thereby avoid expensive simulations at small masses to determine the  $O(a^0)$  and  $O(a)$  pieces of  $\nu(m, \xi)$ .

The complete anisotropic Wilson action con-

sists of the quark action described above and the anisotropic Wilson gauge action (see e.g. [7]), which is parameterized by  $\beta$  and the bare anisotropy  $\xi_0$ . For given  $\beta$ ,  $\xi$  and  $m_0$ , tuning the full action involves determining  $\xi_0$ ,  $\nu$ ,  $\omega_0$  and  $\omega$  self-consistently to give the same renormalized anisotropy  $\xi$  in the gauge and quark sectors, and no  $O(a^0, a)$  errors. Although not a problem of principle, in full QCD this is a costly tuning problem, so we will here only consider the *quenched* case. Then the relation  $\xi_0 = \xi_0(\xi, \beta)$  can be determined once and for all in the pure gauge theory, which we did to high precision in [7].

To non-perturbatively tune the quark action one can use the following strategy: (i) Determine  $\nu(m, \xi)$  by demanding that the pseudo-scalar have a relativistic dispersion relation for  $p \rightarrow 0$ . (ii) Determine the clover coefficients using the Schrödinger functional with the PCAC relation as improvement condition (for details cf. [3–6]). For  $\omega_0$  impose fixed boundary conditions in the time direction to generate a color-electric background field; similarly, for  $\omega$  impose fixed boundary conditions in a spatial direction.

On the classical level this strategy completely decouples the tuning of the three coefficients. On the quantum level this will not hold exactly, and one will have to tune the coefficients iteratively. Since our  $\omega_0$  and  $\omega$  can be chosen to be mass-independent and the mass dependence of  $\nu$  is very weak also on the quantum level for  $\xi \geq 2$  or so (see below), we do not expect serious problems.

The above strategy will require some time and effort to complete. Due to the phenomenological importance and difficulty of the non-perturbative study of heavy quarks we have therefore decided to first present a feasibility study of our approach that does not tune all coefficients non-perturbatively, but only does so for the leading one,  $\nu$ , and uses a tree-level tadpole improvement estimate [8] for  $\omega_0$  and  $\omega$ . For the tadpole estimate we use the mean links in Landau gauge. For the isotropic case it is known [9] that this gives an estimate much closer to the non-perturbative value [5,6] than the mean plaquette estimate.

We performed simulations of the charmonium spectrum at several couplings for  $\xi=3$  and  $\xi=2$ , corresponding to spatial lattice spacings in the

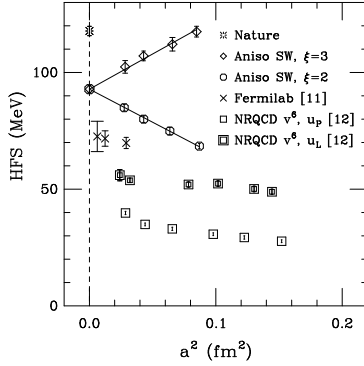


Figure 3. Quenched  $c\bar{c}$  hyper-fine splitting  $1^3S_1 - 1^1S_0$  from our ( $\diamond$ ,  $\circ$ ) and other methods. A joint  $a^2$  fit of our results is shown. For details see [4].

range 0.17 to 0.3 fm (the scale was set very accurately via  $r_0$  [10]). The bare quark mass  $m_0$  and velocity of light  $\nu$  were tuned so that the spin average  $1S$  meson mass equals its observed value and that  $c(0) = 1$  for the pseudo-scalar.  $c(0)$  was obtained by extrapolation from  $c(\mathbf{p})$  at the two lowest on-axis momenta. We find that the  $\nu$  have a rather weak mass dependence; they are even quite close to their classical values of figure 1. We compared the scale from setting  $r_0 = 0.5$  fm with that from the spin-averaged  $1P - 1S$  splitting, finding that they agree within a few percent.

Our main results are the hyper-fine splittings for the  $S$ - and  $P$ -states shown in figures 3 and 4, where they are compared with results from other approaches. Each of our data points uses 400 – 600 configurations with suitably smeared quark sources for early plateaux. We regard our results as the most reliable (note that the strong mass dependence of  $\nu$ ,  $\omega_0$  and  $\omega$  is currently not taken into account in the heavy relativistic approach; also, it had been concluded earlier [12] that NRQCD is breaking down for charmonium, at least for spin splittings). We emphasize the rather non-trivial agreement between the continuum limits of our results for  $\xi = 2$  and 3 in an  $a^2$  continuum extrapolation, which suggests that our estimates of  $\omega_0$ ,  $\omega$  eliminate most  $O(a)$  errors.

To summarize, on-shell  $O(a)$  improvement on anisotropic lattices is a computationally efficient scheme for the study of heavy quarks

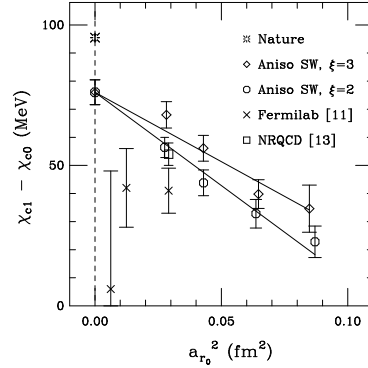


Figure 4.  $P$ -state splitting  $1^3P_1 - 1^3P_0$ .

(all our results were obtained on workstations). Its efficiency is chiefly due to two factors: (i) Anisotropic lattices give very accurate effective masses. (ii) Tuning  $\nu$ , which is the correct way to proceed anyhow, is in practice simpler and more accurate than using the “kinetic mass prescription” as in other approaches. Probably most important, however, it seems that systematic errors due to mass-dependent, relativistic, and quantum corrections to the coefficients in the action can be more easily eliminated on anisotropic lattices [4].

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